8 Discrete Random Variables

D0 Intuitively, to tell whether a random variable is discrete, we simply consider the possible values of the random variable. If the random variable can be limited to only a finite or countably infinite number of possibilities, then it is discrete.

Example 8.1. Voice Lines: A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. [15, Ex 3-1]

D1 **Definition 8.2.** A random variable X is said to be a ***discrete random variable*** if there exists a countable number of distinct real numbers x_k such that

$$\sum_k P[X = x_k] = 1. \tag{13}$$

D2

In other words, X is a discrete random variable if and only if X has a countable support.

Example 8.3. For the random variable N in Example 7.8 (Three Coin Tosses),

The possible values are $0, 1, 2, 3$. (n_1, n_2, n_3, n_4)

The collection of possible values $\{0, 1, 2, 3\}$ is finite. So, the RV is discrete.

For the random variable S in Example 7.9 (Sum of Two Dice),

The possible values are $2, 3, \ldots, 12$. ($s_1, s_2, \ldots, s_{11}$)

The collection of possible values $\{2, 3, \ldots, 12\}$ is finite. So, the RV is discrete.

Example 8.4. Toss a coin until you get a H. Let N be the number of times that you have to toss the coin.

The possible values are $1, 2, 3, \ldots$

The collection of possible values $\{1, 2, 3, \ldots\}$ is countably infinite. So, the RV is discrete.

Example 8.5. Measure the current room temperature.

The possible values are any real numbers between 273.15 to $\approx 1.417 \times 10^{32}$ °C. Any interval of positive length has uncountably many members in it. So, this random variable is *not* discrete.

Set	Signal	RV
countable	digital	discrete
uncountable	analog	continuous

"default" support for discrete RV

8.6. Although the support S_X of a random variable X is defined as any set S such that $P[X \in S] = 1$. For discrete random variable, S_X is usually set to be $\{x : P[X = x] > 0\}$, the set of all "possible values" of X.

Definition 8.7. An ***integer-valued random variable*** is a discrete random variable whose x_k in (13) above are all integers.

x	$P[X=x]$
x_1	0.25
x_2	0.1
x_3	0.01
⋮	

8.8. Recall, from 7.21, that the ***probability distribution*** of a random variable X is a description of the probabilities associated with X. For a discrete random variable, the distribution can be described by just a list of all its possible values $(x_1, x_2, x_3, \ldots)$ along with the probability of each:

$$(P[X = x_1], P[X = x_2], P[X = x_3], \ldots,).$$

In many cases, it is convenient to express the probability in the form of a formula. This is especially useful when dealing with a random variable that has infinite support. It would be tedious to list all the possible values and the corresponding probabilities.

8.1 PMF: Probability Mass Function

Definition 8.9. When X is a discrete random variable satisfying (13), we define its ***probability mass function*** (pmf) by[32]

lowercase

$$p_X(x) = P[X = x].$$

subscript indicates the name of the RV

- Sometimes, when we only deal with one random variable or when it is clear which random variable the pmf is associated with, we write $p(x)$ or p_x instead of $p_X(x)$.
- The argument (x) of a pmf ranges over *all real numbers*. Hence, the pmf is (and should be) defined for x that is not among the x_k in (13) as well. In such case, the pmf is simply 0. This is usually expressed as "$p_X(x) = 0$, otherwise" when we specify a pmf for a particular random variable.

[32]Many references (including [15] and `MATLAB`) does not distinguish the pmf from another function called the probability density function (pdf). These references use the function $f_X(x)$ to represent both pmf and pdf. We will *NOT* use $f_X(x)$ for pmf. Later, we will define $f_X(x)$ as a probability density function which will be used primarily for another type of random variable (continuous RV).

- The pmf of a discrete random variable X is usually referred to as its ***distribution***.

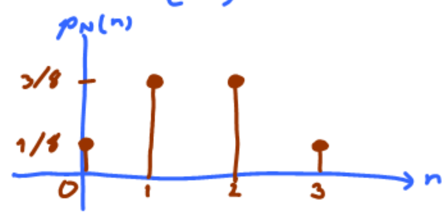
Example 8.10. Continue from Example 7.8. N is the number of heads in a sequence of three coin tosses.

n	$P[N = n]$	
0	1/8	$\longrightarrow p_N(0) = \frac{1}{8}$
1 (TTH, THT, HTT)	3/8	$\longrightarrow p_N(1) = 3/8$
2	3/8	$\longrightarrow p_N(2) = 3/8$
3	1/8	$\longrightarrow p_N(3) = 1/8$

$p_N(0.5) \equiv P[N = 0.5] = 0$

pmf

$$p_N(n) = \begin{cases} 1/8, & n = 0, 3, \\ 3/8, & n = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$



8.11. Graphical Description of the Probability Distribution: Traditionally, we use ***stem plot*** to visualize p_X. To do this, we graph a pmf by marking on the horizontal axis each value with nonzero probability and drawing a vertical bar with length proportional to the probability.

8.12. Any pmf $p(\cdot)$ satisfies two properties:

(a) $p(\cdot) \geq 0$

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x. "$\Sigma = 1$"

When you are asked to verify that a function is a pmf, check these two properties.

Example 8.13. Continue from Example 7.7 where we considered an experiment of rolling one fair dice and Example 7.20 where we defined let $Y(\omega) = (\omega - 3)^2$. Is Y a discrete random variable? If so, find its probability mass function. $p_Y(y) = P[Y = y]$

Solution: First, recall that, in Example 7.20, we plugged in each possible value of ω to find the possible values of Y.

ω	1	2	3	4	5	6
$Y(\omega)$	4	1	0	1	4	9

Y is a discrete random variable because the number of its possible values is four (0,1,4,9) which is finite. Alternatively, (also from Example 7.20), because the finite set $\{0, 1, 4, 9\}$ is a support of Y, we can make the same conclusion that Y is a discrete random variable.

Now, to find its probability mass function $p_Y(y)$, we need to find the probability of each possible value of Y. This can be done using the technique studied in Chapter 7.

- $P[Y = 0] = P(\{3\}) = \frac{1}{6}$.
- $P[Y = 1] = P(\{2, 4\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.
- $P[Y = 4] = P(\{2, 5\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.
- $P[Y = 9] = P(\{6\}) = \frac{1}{6}$.

(Note that for other values y of Y, we have $P[Y = y] = P(\emptyset) = 0$. So, there is no need to consider them.)

$$p_Y(y) = \begin{cases} \frac{1}{6}, & y = 0, \\ \frac{1}{3}, & y = 1, \\ \frac{1}{3}, & y = 4, \\ \frac{1}{6}, & y = 9, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{6}, & y = 0, 9, \\ \frac{1}{3}, & y = 1, 4, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{6}, & y \in \{0, 9\}, \\ \frac{1}{3}, & y \in \{1, 4\}, \\ 0 & \text{otherwise.} \end{cases}$$

8.14. Finding probability from pmf:

As mentioned at the end of Chapter 7, once the random variables are defined, we usually skip mentioning the sample space and the associated "(ω)"-part of the RV. So, we must be able to talk about probability directly from the RV itself. This can be done easily via the pmf.

For "any" subset B of $\mathbb{R}$, we can find

$$P[X \in B] = \sum_{x_k \in B} P[X = x_k] = \sum_{x_k \in B} p_X(x_k).$$

In particular, for integer-valued random variables,

$$P[X \in B] = \sum_{k \in B} P[X = k] = \sum_{k \in B} p_X(k).$$

8.15. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfy the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

Example 8.16. Back to Example 7.7 where we roll one dice.

- The "important" probabilities are

$$P[X=1]=P[X=2]=\cdots=P[X=6]=\frac{1}{6}$$

- In tabular form:

Dummy variable

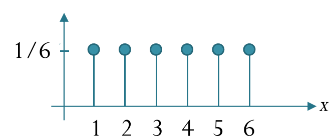
x	$P[X=x]$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

- **Probability mass function (PMF):**

$$p_X(x)=\begin{cases}1/6, & x=1,2,3,4,5,6,\\ 0, & \text{otherwise.}\end{cases}$$

- In general, $p_X(x)\equiv P[X=x]$

- Stem plot:

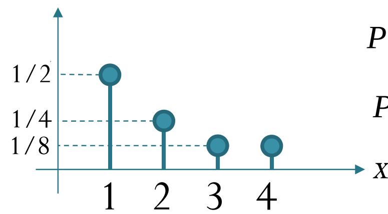


Suppose we want to find $P[X>4]$.

	Steps	For this example...
(a)	Find the support of X.	The support of X is $\{1,2,3,4,5,6\}$.
(b)	Consider only the x inside the support. Find all values of x that satisfy the condition(s).	The members which satisfies the condition ">4" is 5 and 6.
(c)	Evaluate the pmf at x found in the previous step.	The pmf values at 5 and 6 are all 1/6.
(d)	Add the pmf values from the previous step.	Adding the pmf values gives $2/6=1/3$.

Example 8.17. Consider a RV X whose $p_X(x) = \begin{cases} {}^1\!/\!_2, & x = 1, \\ {}^1\!/\!_4, & x = 2, \\ {}^1\!/\!_8, & x \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$

stem plot:



$$P[X=2] = p_X(2) = \frac{1}{4}$$

$$P[X>1] = p_X(2) + p_X(3) + p_X(4) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

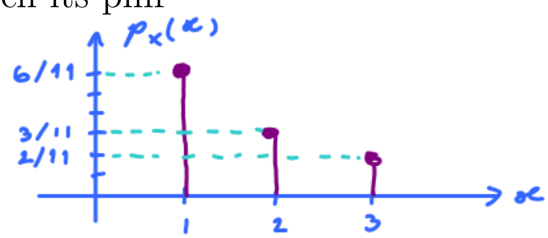
Example 8.18. Suppose a random variable X has pmf

$$p_X(x) = \begin{cases} {}^c\!/\!_x, & x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} {}^6\!/\!_{11x}, & x = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

(a) The value of the constant c is

"$\Sigma = 1$": $\frac{c}{1} + \frac{c}{2} + \frac{c}{3} = 1 \Rightarrow c = \frac{1}{1 + \frac{1}{2} + \frac{1}{3}} = \frac{6}{11}$

(b) Sketch its pmf



(c) $P[X = 1]$

$= p_X(1) = \frac{6}{11}$

(d) $P[X \geq 2] = p_X(2) + p_X(3) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}$.

There are two $\overset{2,3}{}$ x values (in the support) that satisfy the statement "$x \geq 2$"

(e) $P[X > 3] = 0$

None of the x values (in the support) satisfies the statement "$x > 3$".

8.19. Any function $p(\cdot)$ on $\mathbb{R}$ which satisfies

(a) $p(\cdot) \geq 0$, and

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x

is a pmf of some discrete random variable.

8.2 CDF: Cumulative Distribution Function

Definition 8.20. The ***(cumulative) distribution function (cdf)*** of a random variable X is the function $F_X(x)$ defined by

$F_X(5) = P[X \leq 5]$

$F_X(\pi) = P[X \leq \pi]$

$$F_X(x) = P[X \leq x].$$

- The argument (x) of a cdf ranges over all real numbers.
- From its definition, we know that $0 \leq F_X \leq 1$.
- Think of it as a function that collects the "probability mass" from $-\infty$ up to the point x.

8.21. From pmf to cdf: In general, for any discrete random variable with possible values $x_1, x_2, \ldots$, the cdf of X is given by

$$F_X(x) = P[X \leq x] = \sum_{x_k \leq x} p_X(x_k).$$

Example 8.22. Continue from Examples 7.8, 7.12, and 8.10 where N is defined as the number of heads in a sequence of three coin tosses. We have

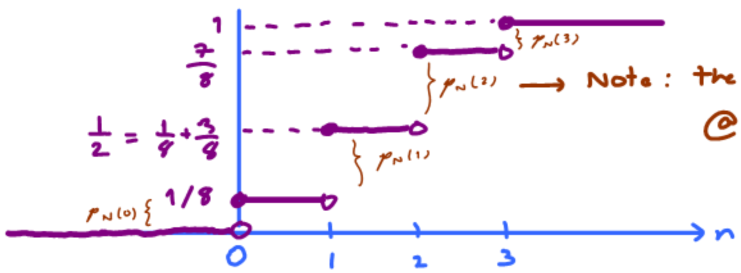
$$p_N(0) = p_N(3) = \frac{1}{8} \text{ and } p_N(1) = p_N(2) = \frac{3}{8}.$$

(a) $F_N(0)$ $\equiv P[N \leq 0] = p_N(0) = \frac{1}{8}$

(b) $F_N(1.5)$ $\equiv P[N \leq 1.5] = p_N(0) + p_N(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

$F_N(n)$

"pmf → cdf" (c) Sketch ~~of~~ its cdf

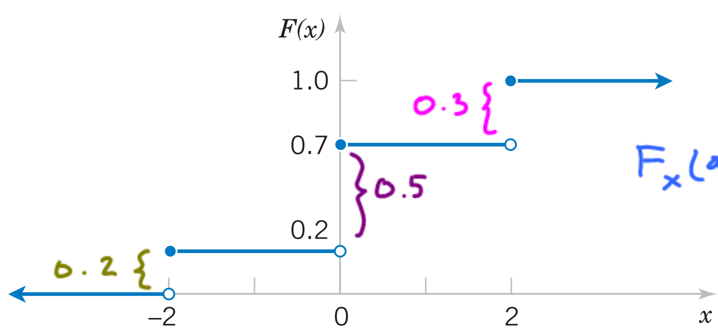


→ Note: the amount of jump @ $n=2$ is $p_N(2) = \frac{3}{8}$

8.23. Facts:

- For any discrete r.v. X, F_X is a right-continuous, ***staircase*** function of x with jumps at a countable set of points x_k.
- When you are given the cdf of a discrete random variable, you can derive its pmf from the locations and sizes of the jumps. If a jump happens at $x = c$, then $p_X(c)$ is the same as the amount of jump at c. At the location x where there is no jump, $p_X(x) = 0$.

"cdf → pmf" **Example 8.24.** Consider a discrete random variable X whose cdf $F_X(x)$ is shown in Figure 19.

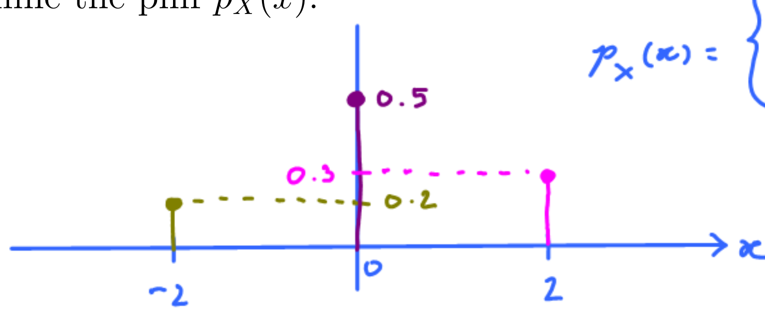


$$F_X(x) = \begin{cases} 0, & x < -2, \\ 0.2, & -2 \le x < 0 \\ 0.7, & 0 \le x < 2 \\ 1, & x \ge 2. \end{cases}$$

Figure 19: CDF for Example 8.24

Determine the pmf $p_X(x)$.

$$p_X(x) = \begin{cases} 0.2, & x = -2, \\ 0.5, & x = 0, \\ 0.3, & x = 2, \\ 0, & \text{otherwise}. \end{cases}$$



ECS 315: In-Class Exercise # 13 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. [ENRE] Explanation is not required for this exercise.
3. **Do not panic.**

Date: 1 5 / 1 0 / 2019			
Name	ID (last 3 digits)		
Prapun	5	5	5

1. Consider a random variable X whose pmf is given by

$$p_X(x) = \begin{cases} 0.2, & x = -1, \\ c, & x = 1,3, \\ 0, & \text{otherwise.} \end{cases}$$

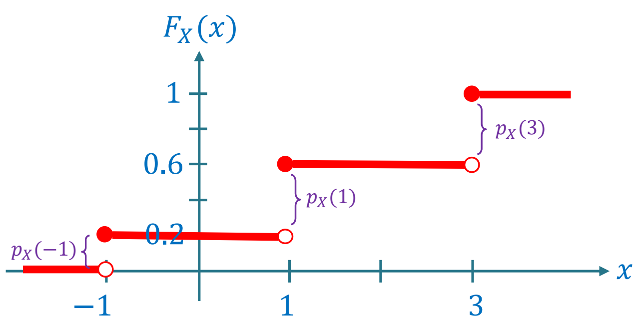
a. Find the constant c.

"$\Sigma = 1$" $\Rightarrow p_X(-1) + p_X(1) + p_X(3) = 1$

$$0.2 + c + c = 1$$

$$c = 0.4$$

b. Plot the cdf of this random variable.



Recall that the cdf can be derived from the pmf by using the $p_X(x)$ as the jump amount at x.

2. Consider a random variable X whose cdf is given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.2, & 0 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

At $x = 0$, there is a jump of size 0.2.

At $x = 3$, there is a jump of size 0.8.

a. Find $P[X \le 1]$.

By definition, $P[X \le 1] = F_X(1) = 0.2$.

b. Find $P[X > 1]$.

Because $[X > 1]$ and $[X \le 1]$ are opposite (complementary) events, we know that

$$P[X > 1] = 1 - P[X \le 1] = 1 - 0.2 = 0.8.$$

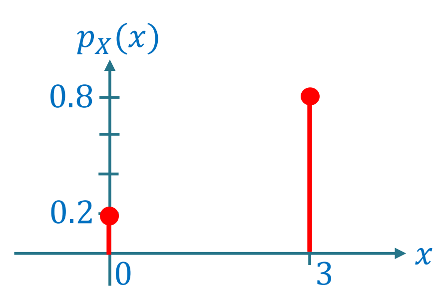
c. Plot the pmf of X.

For discrete RV, the pmf can be derived from the jump amounts in the cdf plot.

Here, the jumps in the cdf happen twice: at $x = 0$ and $x = 3$.

The jump amounts are 0.2 and 0.8, respectively.

Therefore, $p_X(x) = \begin{cases} 0.2, & x = 0, \\ 0.8, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$



(true for any RV)

8.25. Characterizing[33] properties of cdf:

CDF1 F_X is non-decreasing (monotone increasing)

$\equiv$ if $a < b$, then $F_X(a) \le F_X(b)$

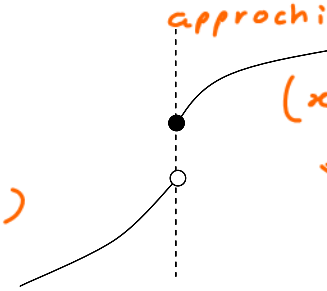
CDF2 F_X is right-continuous (continuous from the right)

$F_X(x^+) = F_X(x)$

Right-handed limit

The limit as x decreases in value approching "c".

$$F_X(c^+) = \lim_{x \searrow c} F_X(x) = \lim_{\substack{h \to 0 \\ h > 0}} F_X(c+h)$$



(x approaches c "from the right" or "from above")

Figure 20: Right-continuous function at jump point

CDF3 $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

8.26. For discrete random variable, the cdf F_X can be written as

$$F_X(x) = \sum_{x_k} p_X(x_k) u(x - x_k),$$

where $u(x) = 1_{[0,\infty)}(x)$ is the unit step function.

[33]These properties hold for any type of random variables. Moreover, for any function F that satisfies these three properties, there exists a random variable X whose CDF is F.